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Newtonian Gravitational Field Theory

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Abstract

A field theory for gravitation is developed within the framework of the special theory of relativity. This is achieved by exploiting the similarity in mathematical structure of two relations which are found in both Newton's gravitational theory and Maxwell's electromagnetic theory. These relations are: (1) the law of force between the relevant physical entities (mass and electric charge), and (2) the equation of continuity (conservation of charge). The field equations describe the propagation of gravitational waves with the velocity of light in much the same way that Maxwell's field equations describe electromagnetic waves. Both fields have such similar mathematical structures that they are developed in parallel up to the point where their inherently different physical content cause their paths of evolution to diverge. At this stage, the field equations for both theories are determined. The physical significance of the field variables of both theories imposes a mathematical formalism which does *not* give rise to self-interactions. A calculation for the energy in the field of two particles representative of either the electromagnetic or gravitational field is shown to give the correct finite value. The reason that conventional calculations yield an infinite energy is readily seen to lie in the calculation of a physically meaningless quantity. The mathematical formalism required by the field theories is used to develop generalizations of the usual conservation laws. Two conservation laws are derived which are consequences of the consistent physical interpretation of the field variables. These laws do not appear in conventional theory. The approach followed here in developing the field theories leads to the appearance of forces dual to the well-known forces. Thus, for the electromagnetic field, we find a dual to the Lorentz force and, in the gravitational field, we find a dual to Newton's law of gravitation. These results are not due to the introduction of the fields, for they can be expressed in terms of the particle variables. They emerge from the consistent application of the physical interpretation of the particle and field variables. A basic physical principle, which underlies both theories, is best expressed by the statement: It is the interactions between the elements of a physical event and not the elements themselves which are the physical observables.

1. *Introduction*

Newtonian gravitational theory has long been regarded as the prototype of all action-at-a-distance theories. Maxwell's electromagnetic theory occupies the same position in regard to field theories. Both theories share a number of common mathematical elements, and we will exploit this overlapping area to develop in parallel a field theory for each. We will find striking similarities between the field theories as they take shape, until we reach a point where the inherently different physical content of the two theories causes their paths of development to diverge.

The mathematical formulations of the two theories have two relations which can, with a change in notation, be made identical in form. These are: (1) Newton's law of gravitation and Coulomb's law of force, and (2) the continuity equation, or—as they would be referred to in their respective domain—the conservation of mass and the conservation of charge.

We shall adopt a notation which will enable us to develop both field theories in parallel. Such a procedure will avoid repetition and will have the appeal of freshness and novelty. The transcription from the new notation to the corresponding field variables E and H will be immediate. The notation can then be used exclusively for the gravitational field equations.

Our first concern will be the definition of the field variables. A careful analysis of the physical significance of the field variables will impose certain criteria on their mathematical representation and application. The interpretation of the field variables will also determine the mathematical formalism that must be used to represent the physics of the situation.

The next consideration will be an examination of the role of the continuity equation. It is at this stage in the parallel development of the theories that the distinctive physical content of each theory requires their paths of mathematical evolution to diverge. At this point, the field equations will be determined.

There will still remain a formal similarity between the two theories, which will be useful in deriving conservation laws. Both fields will give rise to conservation laws, which are a consequence of the mathematical formalism imposed by the introduction of the field variables. For the gravitational field, all of these relations are presented for the first time. In the case of the electromagnetic field, they have appeared in abbreviated form, but a more detailed report will be submitted for publication. To make this report self-contained, I will include the material we need.

The features of the electromagnetic theory, which are present in our interpretation but not in conventional theory, are the components of a four-vector which we have designated as the dual of the Lorentz force. The gravitational field equations mirror these relations, so that we find a dual to the four-vector representation of Newton's gravitational law. I have shown (Schwebel, 1970) that the dual Lorentz force plays a significant role in the motion of charged particles. One aspect is the precession of the perihelion of the orbit of one charged particle about an oppositely charged particle. The gravitational dual of Newton's law reflects the same property, and a detailed report covering these gravitational aspects will soon be made available.

Finally, I establish for both fields the close connection between the representation of the source by a particle model and its equivalent representation by a field model. The particle-wave dualism is shown to be merely two different mathematical representations of the same physical entity.

2. *Newton's Law of Gravitation and Coulomb's Law*

The introduction of field variables is accomplished by treating Newton's law of gravitation as we did Coulomb's law in defining the intensity of the electric field. To pursue the developments of both fields simultaneously, we will write Newton's law of gravitation,

$$\mathbf{F} = \frac{-Gm_1 m_2 \mathbf{r}}{r^3} \tag{2.1}$$

in the form which makes it mathematically identical to Coulomb's law of force between electric charges.

$$F = \frac{\mu_1 \mu_2 \mathbf{r}}{r^3}$$

where

$$\mu = (-G)^{1/2} m$$

The introduction of μ is an obvious convenience, so that we formally can talk about it as the source of either field without having to continually adjust our language to include the gravitational constant G and the minus sign in equation (2.1).

We define the Newtonian (or electric) field intensity, N, due to the source μ by the well-known relation

$$\mathbf{N} = \frac{\mu \mathbf{r}}{r^3} \tag{2.2}$$

The two differential equations which are a consequence of equation (2.2) are

$$\nabla . \mathrm{N} = 4\pi\rho \tag{2.3}$$
$$\nabla \times \mathrm{N} = 0$$

where ρ is the source density defined by the equation

$$\mu = \int \rho \, d\tau$$

in which $d\tau$ is a volume element, and the integration is over all of space.

So far the procedure and results are standard. In equation (2.3), we can recognize electrostatics and gravitational potential theory. There is, however, an important remark to be made about the definition of N given in equation (2.2) and its relation to equations (2.1) and (2.3). The field variable N is a *replacement* for the source μ. Equation (2.2) is a mathematical device for attaching to each point about the source μ a vector N with its direction and magnitude determined by equation (2.2). When we bring a second source, μ', into the field, our knowledge of the value of N at the position of μ' enables us to calculate the force on it without reference to μ, the source of N. N *replaces* the need to refer to μ to obtain the force on μ' from equation (2.1). This mathematical technique introduces the field concept in distinction to the action-at-a-distance formalism that equation (2.1) represents. At the same time, the *definition* of **N** *excludes* the formation of any mathematical term of the form $\mu\mathbf{N}$, where μ is the source of **N**. The exclusion is on the basis that such a term has no physical meaning according to equations (2.1) and (2.2). We must bear in mind that **r** is the displacement between the source and the charge brought into its field.

Because terms like $\mu\mathbf{N}$ must be excluded, we identify each field variable with its source. Thus, equations (2.1), (2.2) and (2.3) should be written

$$\mathbf{F} = \mu^p \mathbf{N}^q$$
$$\mathbf{N}^q = \frac{\mu^q \mathbf{r}}{r^3} \tag{2.4}$$

and

$$\nabla \cdot \mathbf{N}^q = 4\pi\rho^q \tag{2.5a}$$

$$\nabla \times \mathbf{N}^q = 0 \tag{2.5b}$$

The labeling by the use of superscripts will insure the exclusion of terms which represent a self-interaction—such as a field interacting with its own source.

The definition of the field variables leads to further limitations. If there are no sources, then there cannot be corresponding field variables and the converse is true. One consequence is that the solutions to the homogeneous equations associated with equations (2.5a) and (2.5b) must be rejected—they are not physically acceptable. But, there are such solutions so that we must devise a method which will determine only the desired type of solution. A mathematical technique that will accomplish the designated task will be given.

It is readily shown that a solution under the stated conditions must be unique. For, if there were two such solutions, then their difference would be a solution of the homogeneous set of equations (the operators are linear). Unless the two postulated solutions are identical, one of them would not satisfy the initial assumption. Thus, the uniqueness of the solution has been established. Therefore, we see that the introduction of the field variables and their physical interpretation have determined the mathematical formalism that must be developed to represent them.

A corollary to what we have just discussed is that the field equations developed to this point are in fact tautological; the left-hand sides of these equations are just another mathematical formulation of the right-hand sides. The observation that both equation (2.5a) and (2.5b) stem from the definition (2.2) is proof of that. It is worthwhile to emphasize the tautological aspect of the field equations. Thus, if we regard the right-hand side as a particle description of the source, then the left-hand side of equations (2.5a) and (2.5b) is an equivalent field description.

The procedure we have followed is identical to that which is pursued in developing electrostatics. The identification of **N** with **E** will readily come to the reader's mind. Potential theory is similarly treated but the symbols are different. Thus, we would replace **N** by $\mathbf{N} = \nabla V$ and obtain Poisson's equation relating the potential function, V, to the distribution of mass. Our presentation differs from the conventional one in the stress we place on the consistent application of the concept of the field variable. Some of the consequences of this approach have been explored and others depend on the determination of the field equations for both of the theories under study. We now turn to the completion of that task.

3. *The Equations of Continuity*

One of the basic relations in mechanics and electricity is the equation of continuity. In the former it is called the conservation of mass, and in the latter it is known as the conservation of charge. Both equations have the same mathematical form, which in an obvious notation (in conformity with our stated purpose) can be written as

$$\partial_t \rho^p + \nabla . (\rho^p \mathbf{v}^p) = 0. \tag{3.1}$$

We may look upon this equation as a new starting point—though clearly we will be motivated by equations (2.5a) and (2.5b)—and define the density by the relation

$$\rho^p = (\tfrac{1}{4}\pi) \nabla . \mathbf{N}^p \tag{3.2}$$

The definition of the density, ρ^p, implies the same relation between ρ^p and $\mathbf{N}^p$ as before. Namely, when $\rho^p = \rho^p(xyzt)$ vanishes throughout all space-time so must $\mathbf{N}^p$. The converse is obviously true. Again, terms like ρ^p, $\mathbf{N}^p$, though mathematically valid, are not acceptable as representing a meaningful physical quantity. Hence, this approach leads to the same conclusions we had reached earlier.

Equation (3.2) substituted into equation (3.1) yields the result that

$$\nabla . (\partial_t \mathbf{N}^p + 4\pi \rho^p \mathbf{v}^p) = 0 \tag{3.3}$$

Consequently, we can define a field variable, $\mathbf{M}^p$, such that

$$\partial_t \mathbf{N}^p + 4\pi \rho^p \mathbf{v}^p = \nabla \times \mathbf{M}^p \tag{3.4}$$

We have omitted the dependence of the source and field variables on the space-time coordinates solely for the purpose of concentrating on the derivations of the field equations without being troubled by irrelevant details.

If in equation (3.4), **N** is replaced by **E** and **M** by **H**, we obtain one of Maxwell's electromagnetic equations.

Should the source term $\rho^p \mathbf{v}^p$ vanish, then so must $\mathbf{M}^p$. For, if $\rho^p \mathbf{v}^p$, i.e., the current $\mathbf{J}^p$, vanishes, then the equation of continuity reduces to $\partial_t \rho^p = 0$, and we have the theories reduced to the area of statics which is governed by equations (2.5a) and (2.5b). The absence of a current $\mathbf{J}^p$ implies the absence of the field variable $\mathbf{M}^p$. But the question whether the existence of $\mathbf{J}^p = \rho^p \mathbf{v}^p$ implies the existence of the field variable $\mathbf{M}^p$ remains open. For the electromagnetic field, the answer is undeniably in the affirmative. Oersted's experiment gives us an unequivocal answer and we know that $\mathbf{M} = \mathbf{H}$ is a meaningful physical field variable.

In gravitational theory, we have no such experimental result. Nor, do we have the counterpart of Faraday's induction experiment, which generalizes the second of equations (2.5a) and (2.5b) to include a time derivative of the field **M**. These considerations weigh the balance in favor of the conclusion that in the gravitational field there is no field variable **M**. The conclusion, however, is based on the absence of certain results which do not preclude the existence of a field variable like **M**. One can argue that such a field does exist but that its value is so small that it has not been observed—to which could be added that little has been done to detect such fields (Dicke, 1962). We will present a theoretical argument for assuming **M** to be absent in gravitational field theory. For the moment, let us assume that **M** does exist.

The introduction of the field variable **M** is, as we have seen, dependent upon the existence of a current, $\mathbf{J} = \rho \mathbf{v}$. If there are no currents, then there cannot be corresponding field variables **M**. Thus, there are no independent sources for these field variables, and we can conclude that the relation

$$\nabla . \mathbf{M}^p = 0 \tag{3.5}$$

is valid.

The final relation is one which generalizes equation (2.5b) and is a direct consequence of Faraday's induction experiment. It is the well-known relation given below as equation (3.6c)

Collecting all the field equations we have

$$\nabla . \mathbf{N}^p = 4\pi\rho^p \tag{3.6a}$$

$$\nabla . \mathbf{M}^p = 0 \tag{3.6b}$$

$$\nabla \times \mathbf{N}^p + \partial_t \mathbf{M}^p = 0 \tag{3.6c}$$

$$\nabla \times \mathbf{M}^p - \partial_t \mathbf{N}^p = 4\pi\rho^p \mathbf{v}^p \tag{3.6d}$$

The Lorentz invariance of the equation of continuity assures us that the velocity of propagation of the effects of both fields proceeds with the speed of light.

Equations (3.6a)–(3.6d) with both field variables **N** and **M** represent Maxwell's electromagnetic equations. We have questioned the representation of the gravitational field by two field variables by indicating that the evidence points to the need for only one field variable, **N**, to represent the gravitational field. We wish to support this conclusion with a theoretical argument.

Let us assume that if the current $\mathbf{J}^p = \rho^p \mathbf{v}^p$ does not vanish, then $\mathbf{M}^p$ must not vanish. For the gravitational field, the current represents the momentum density. In the special instance in which it is constant, Newton's laws of motion imply that no force density should be attributed to the constant current. For according to Newton's third law, if it exerts a force via the field variable **M**, then there is a force exerted on the current. Since the constancy of the current contradicts the existence of the force of reaction, we must conclude that the force of action was nil and that **M** is absent.

We have seen that if a source current is zero, then **M** is zero, and now we have established that the presence of a constant source current does not require that **M** be present. The argument for the non-existence of the field variable **M** for the gravitational field is strong indeed, but not conclusive. We have eliminated the possibility of a gravitational experiment mirroring the Oersted experiment for the electromagnetic field, but we have not established the absence of a counterpart to Faraday's induction experiment. Moreover, we may well call into the arena for critical examination Newton's laws of motion.

Since the special theory of relativity does not alter the fundamental significance of Newton's laws—rate of change of momentum still implies the presence of a force though both of these concepts are generalized—then we may still maintain the absence of the field variable **M** from gravitational field theory for the larger domain of relativistic mechanics. But, the generalization of Newtonian mechanics that is furnished by quantum theory may well require a reexamination of the role of **M** in gravitational field theory. At present, within the framework of classical relativistic mechanics, the weight of the evidence favours the absence of **M**. Within the limits we have outlined, we will find the choice for **M** given added support after we have derived a number of conservation laws.

4. *Conservation Laws*

It will be convenient to establish the conservation laws in general, i.e., without setting $\mathbf{M} = 0$, for that value can be inserted later to obtain the relations pertinent to the gravitational field.

To derive the equation for the conservation of energy, we make use of equations (3.6c) and (3.6d) to obtain—in an obvious mathematical procedure—the relations

$$\mathbf{M}^q . \partial_t \mathbf{M}^p + \mathbf{M}^q . \nabla \times \mathbf{N}^p = 0$$

$$\mathbf{N}^q . \partial_t \mathbf{N}^p - \mathbf{N}^q . \nabla \times \mathbf{M}^p = -4\pi\rho^p \mathbf{v}^p . \mathbf{N}^q$$

Note that the interaction terms are between field variables whose sources are distinct. To the above equations, we add the two additional equations obtained by interchanging the superscripts p and q. The result we find to be

$$(\tfrac{1}{4}\pi)\, \partial_t \{\mathbf{N}^p . \mathbf{N}^q + \mathbf{M}^p . \mathbf{M}^q\} + (\tfrac{1}{4}\pi) \nabla . \{\mathbf{N}^p \times \mathbf{M}^q + \mathbf{N}^q \times \mathbf{M}^p\}$$
$$= -\{\rho^p \mathbf{v}^p . \mathbf{N}^q + \rho^q \mathbf{v}^q . \mathbf{N}^p\} \quad (4.1)$$

in which we have used the vector identity

$$\nabla . (\mathbf{A} \times \mathbf{B}) = \mathbf{B} . \nabla \times \mathbf{A} - \mathbf{A} . \nabla \times \mathbf{B}$$

Equation (4.1) is the conservation law we set out to establish. If we identify **N** with **E** and **M** with **H**, we obtain a generalization of the conservation law which is derived in conventional electromagnetic theory. If we drop the distinction which the superscripts p and q denote, then the above equation becomes identical with the conventional result.

The importance of the identification of the field variable with its source is clearly demonstrated by carrying out the calculation for the energy in the field of two stationary sources separated by a distance d. The energy density, w, is given by the defining equation,

$$w = (\tfrac{1}{4}\pi)\{\mathbf{N}^p.\mathbf{N}^q + \mathbf{M}^p.\mathbf{M}^q\} \tag{4.2}$$

For the physical situation we are considering, we have $\mathbf{N}^p = \mu^p \mathbf{r}^p / r^{p^3}$, $\mathbf{N}^q = \mu^q \mathbf{r}^q / r^{q^3}$ and $\mathbf{M}^p = \mathbf{M}^q = 0$. The result will be valid for the two field theories. Inserting these values into equation (4.2), we perform an elementary integration to find that the energy in the field is

$$\frac{\mu^p \mu^q}{d}$$

a not unexpected result.

The conventional calculation proceeds differently. The energy is given by

$$I = (\tfrac{1}{8}\pi) \int d\tau \, \mathbf{N}.\mathbf{N}$$

in which

$$\mathbf{N} = \mathbf{N}^p + \mathbf{N}^q$$

We find that

$$I = (\tfrac{1}{8}\pi) \int d\tau \{\mathbf{N}^p.\mathbf{N}^p + \mathbf{N}^q.\mathbf{N}^q + 2\mathbf{N}^p.\mathbf{N}^q\}$$

The first two terms in the integrand are the ones which give rise to the infinite self-energy terms; the last term is the correct expression. We see why a 'subtraction' procedure must be used to make this approach 'work'. The origin of the difficulty is clear; the self-energy terms like $\mathbf{N}^p.\mathbf{N}^p$ which are physically meaningless have been introduced inadvertently by the mathematical procedure. The identification of source and corresponding field variable which we have employed prevents the appearance of physically meaningless mathematical forms. Notice that if we perform the calculation for a single particle, then the conventional procedure yields an infinite result; whereas, the present interpretation tells us that the calculation has no meaning. For, a single particle which does not interact with another particle or system of particles does no work and hence stores no energy. These considerations are another reflection of the particle-field tautology which was discussed earlier.

The derivation of the equation for the conservation of momentum of the field proceeds along the same lines as in conventional theory, except for slight changes which are required by the labeling of the field variables. We obtain

$$(\tfrac{1}{4}\pi)\,\partial_t\{\mathbf{N}^p \times \mathbf{M}^q + \mathbf{N}^q \times \mathbf{M}^p\} + (\tfrac{1}{4}\pi)\nabla\cdot R$$
$$= -\{\rho^p\mathbf{N}^q + \rho^p\mathbf{v}^p \times \mathbf{M}^q + \rho^q\mathbf{N}^p + \rho^q v^q \times \mathbf{M}^p\} \quad (4.3)$$

in which the components of R are the elements of a tensor. In the electromagnetic field, it is called the Maxwell stress tensor. We have for R_x the components

$$R_{xx} = \mathbf{N}^p\cdot\mathbf{N}^q + \mathbf{M}^p\cdot\mathbf{M}^q - 2N_x{}^p N_x{}^q - 2M_x{}^p M_x{}^q$$
$$R_{xy} = -N_x{}^p N_y{}^q - N_x{}^q N_y{}^p - M_x{}^p M_y{}^q - M_x{}^q M_y{}^p$$
$$R_{xz} = -N_x{}^p N_z{}^q - N_x{}^q N_z{}^p - M_x{}^p M_y{}^q - M_x{}^q M_z{}^p$$

The other components are obtained from R_x by permuting x, y and z.

The right-hand side of equation (4.3) when expressed in electromagnetic field variables represents the sum of Lorentz force densities; one is exerted on p by q and the other is that of p on q. For the gravitational field, the interpretation is different. If the velocities vanish, then, as we have observed previously, **M** must vanish. It follows from equation (4.3) that

$$\int \rho^p\mathbf{N}^q\,d\tau = -\int \rho^q\mathbf{N}^p\,d\tau \quad (4.4)$$

The relation represents the law of action and reaction for the particular situation we have been considering. When the currents are constant, the field variables also vanish, as we have seen. Again, for this case, we have $\mathbf{M} = 0$, and Newton's third law of motion [equation (4.4)] is valid. If we set $\mathbf{M} = 0$, the law of action and reaction will be validly represented by the field variables of the gravitational field. Although the final decision in this matter must be left to experiment, we will not consider **M** a field variable for the gravitational field.

There are two other conservation laws which are uniquely the consequence of our treatment of the field variables and their sources. The first one of these is obtained by following the same procedure used in deriving equation (4.1)—the conservation of energy equation. We use equations (3.6c) and (3.6d) to form the equations

$$\mathbf{N}^q\cdot\partial_t\mathbf{M}^p + \mathbf{N}^q\cdot\nabla \times \mathbf{N}^p = 0$$
$$\mathbf{M}^p\cdot\partial_t\mathbf{N}^q - \mathbf{M}^p\cdot\nabla \times \mathbf{M}^q = -4\pi\rho^q\mathbf{v}^q\cdot\mathbf{M}^p$$

Adding yields,

$$\partial_t(\mathbf{N}^q\cdot\mathbf{M}^p) + \mathbf{N}^q\cdot\nabla \times \mathbf{N}^p - \mathbf{M}^p\cdot\nabla \times \mathbf{M}^q = -4\pi\rho^q\mathbf{v}^q\cdot\mathbf{M}^p$$

Interchange p and q and subtract the resulting equation from the above to find

$$\partial_t\{\mathbf{N}^p\cdot\mathbf{M}^q - \mathbf{N}^q\cdot\mathbf{M}^p\} + \nabla\cdot\{\mathbf{N}^q \times \mathbf{N}^p + \mathbf{M}^p \times \mathbf{M}^q\}$$
$$= -4\pi\{\rho^p\mathbf{v}^p\cdot\mathbf{M}^q - \rho^q\mathbf{v}^q\cdot\mathbf{M}^p\} \quad (4.5)$$

The right-hand side of the equation has the dimensions of an energy density for the electromagnetic field variables. For the gravitational field, the relation reduces to

$$\nabla.(\mathbf{N}^q \times \mathbf{N}^p) = 0 \tag{4.6}$$

which can be derived directly from equation (3.6c). Note that each member in equations (4.5) and (4.6) survives because of the distinction imposed on the field variables by the labels p and q. In other words, the conservation law is a consequence of the consistent mathematical application of the definition of the field variables. The gravitational equation (4.6) can be interpreted very much along the same lines as the Poynting vector of electromagnetic theory as generalized by our presentation. We can support such a view by obtaining from equation (4.1) the gravitational relation

$$(\tfrac{1}{4}\pi)\,\partial_t(\mathbf{N}^p.\mathbf{N}^q) = -\{\rho^p\,\mathbf{v}^p.\mathbf{N}^q + \rho^q\,\mathbf{v}^q.\mathbf{N}^p\} \tag{4.7}$$

The missing divergence term, which is expected in a conservation law, could be said to be equation (4.6). However, such an interpretation is one of form, not of substance, and equation (4.7) as it stands is a conservation law as such laws have been designated in field theory. The present theory interprets these relations as two equivalent mathematical representations of the same physical entity. But, the change in nomenclature would cause difficulties of relating corresponding elements of the present theories and conventional theories so we have made no attempt to alter accepted usage.

Now let us turn to the derivation of the second conservation law which is unique to the present interpretation. To establish it, we pursue the same course used to establish equation (4.3) except for the changes necessitated by a change in the multiplying field variables. Our result is

$$\partial_t\{\mathbf{N}^q \times \mathbf{N}^p + \mathbf{M}^q \times \mathbf{M}^p\} + \nabla.K$$
$$= -4\pi\{\rho^p\,\mathbf{M}^q - \rho^p\,\mathbf{v}^p \times \mathbf{N}^q - \rho^q\,\mathbf{M}^p + \rho^q\,\mathbf{v}^q \times \mathbf{N}^p\} \tag{4.8}$$

with the components of K forming a tensor similar to the tensor R of equation (4.3). The components of K can be determined by replacing $(\mathbf{N}^q, \mathbf{M}^q)$ in corresponding components of R with $(\mathbf{M}^q, -\mathbf{N}^q)$. The right-hand side of equation (4.8) has the dimensions of a force density, and has been named the dual Lorentz force density (Schwebel, 1970) for the electromagnetic field. Some of its contributions to phenomena in classical and quantum electrodynamics have been explored (Sachs & Schwebel, 1961; Mann & Schwebel, 1965). For the gravitational field, equation (4.8) becomes

$$\partial_t(\mathbf{N}^q \times \mathbf{N}^p) = 4\pi\{\rho^p\,\mathbf{v}^p \times \mathbf{N}^q - \rho^q\,\mathbf{v}^q \times \mathbf{N}^p\} \tag{4.9}$$

The right-hand side is the dual force density of the gravitational field. The relation also follows from equation (3.6d) once $\mathbf{M}$ is set equal to zero, but the method of derivation we employed enables us to exhibit the relationship which exists between the two fields—their similarities and differences. Now we turn to study the field of our main concern.

5. *Gravitational Field Theory*

We can now assemble the equations which represent the gravitational field.

$$\nabla \cdot \mathbf{N}^p = 4\pi\rho^p \qquad (5.1a)$$

$$\nabla \times \mathbf{N}^p = 0 \qquad (5.1b)$$

$$\partial_t \mathbf{N}^p = -4\pi\rho^p \mathbf{v}^p \qquad (5.1c)$$

From equations (5.1a) and (5.1c) it follows that

$$\partial_t \rho^p + \nabla \cdot (\rho^p \mathbf{v}^p) = 0$$

From all the equations we can obtain for the static field

$$\mathbf{N}^p = \frac{\mu^p \mathbf{r}^p}{r^{p^3}}$$

where $\mathbf{r}^p$ is the displacement of the field point from the source μ^p. In other words, we can abstract what we have put in. There is an important relation which can be obtained from (5.1b) and (5.1c). The momentum density $\rho^p \mathbf{v}^p$ is irrotational and can be expressed as the gradient of a scalar function

$$\nabla \times (\rho^p \mathbf{v}^p) = 0$$

Two important tasks confront us. The first is to show that equations (5.1a)–(5.1c) are Lorentz covariant. Though it was shown that these equations were obtained from well-known covariant equations by setting **M** equal to zero, it does not follow that the reduced equations remain Lorentz covariant. Equations (5.1a)–(5.1c) are not manifestly covariant. The second task is to present the mathematical technique by which equations (5.1a)–(5.1c) as well as (3.6a)–(3.6d) can be solved so that their solution does not contain solutions of the homogeneous equation. More precisely, we want solutions which are linearly independent of the manifold of solutions of the homogeneous equation.

Both tasks can be performed together. The method of solution rests on the observation that, if we have a linear operator, L, such that

$$LA = B,$$

then we can define uniquely an inverse operator, L^{-1}, provided that the domain of L is properly defined. By the domain of an operator is meant the set of functions to which the application of the operator is restricted. Once the inverse has been obtained, it follows that

$$A = L^{-1} B$$

and that if $B = 0$ then so is A.

For the moment, let us assume that we can find the inverse of the operator. Take the curl of equation (5.1b) and use the other equations [(5.1a) and (5.1c)] to obtain the result,

$$\Box \mathbf{N} \equiv \{\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2\}\mathbf{N} = -4\pi\{\nabla\rho + \partial_t(\rho\mathbf{v})\} \qquad (5.2)$$

We have dispensed with the superscripts, for they serve no useful purpose in the following purely mathematical discussion. The D'Alembertian operator, $\Box$, also arises in the general equations (3.6a)–(3.6d) so that the results we will obtain hold for that set of equations. Another relation we need and can obtain from equations (5.1a)–(5.1c) is the continuity equation

$$\partial_t \rho + \nabla . (\rho \mathbf{v}) = 0 \tag{5.3}$$

The derivations of equations (5.2) and (5.3) imply that the solutions of equations (5.1a)–(5.1c) are also solutions of the derived equations. However, in general the converse is not true. But, if the inverse to the D'Alembertian, $\Box$, exists, we will show that the solution of the derived equation is also the solution of the original set of equations.

Once we have established the equivalence between the two sets of equations, we need only consider one of them and establish the existence of the inverse for that particular operator. Thus, we can shift our attention from equations (3.6a)–(3.6d) and (5.1a)–(5.1c) to equations (5.2) and (5.3).

Let us take the divergence of equation (5.2),

$$\Box \nabla . \mathbf{N} = -4\pi\{\Delta\rho - \partial_t{}^2 \rho\} = 4\pi \Box \rho$$

whence we obtain (5.1a), since $\Box^{-1}$ has been assumed to exist. The continuity equation has been used to obtain the form of the central term. Again from equation (5.2), we obtain

$$\Box \partial_t \mathbf{N} = -4\pi\{-\Delta(\rho \mathbf{v}) + \partial_t{}^2(\rho \mathbf{v})\} = -4\pi \Box (\rho \mathbf{v})$$

and see that the existence of $\Box^{-1}$ yields equation (5.1c). The continuity equation has again been employed to obtain the central term. A third return to equation (5.2) gives us

$$\Box(\nabla \times \mathbf{N}) = -4\pi \partial_t(\nabla \times (\rho \mathbf{v})) = \partial_t{}^2(\nabla \times \mathbf{N})$$

Here we have used the result just obtained, namely, equation (5.1c). Simplifying the equation we find

$$\Delta(\nabla \times \mathbf{N}) = 0$$

We will show that the existence of $\Box^{-1}$ implies that of Δ^{-1}. Hence, it follows that:

$$\nabla \times \mathbf{N} = 0$$

and we have derived the remaining equation (5.1b).

The one-to-one correspondence between the equations (5.1a)–(5.1b) and the equations (5.2) and (5.3) enables us to reduce the analysis of the operators involved to a determination of the inverse of the D'Alembertian. At the same time, we have established a set of manifestly covariant equations which are equivalent to the original set of equations [(5.1a)–(5.1c)].

6. *The Inverse of the D'Alembertian Operator*

The construction of the inverse of the D'Alembertian operator is equivalent to the determination of a particular Green's function, $G(\mathbf{r},t;\mathbf{r}',t')$, which is the solution of the equation

$$\Box G(\mathbf{r},t;\mathbf{r}',t') = 4\pi\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \tag{6.1}$$

In general, there are an unlimited number of solutions to this equation, each determined by special boundary conditions. The difference between any two such solutions is a solution to the homogeneous equation. Thus, we see that the condition that no solution to the homogeneous equation be any part of the physically acceptable solution is sufficiently strong to eliminate all but one of the infinite number of possibilities.

To select the one Green's function we want, we proceed as follows. We take the Fourier transform over the space variables and a one-sided Laplace transform over the time variable of the equation for the Green's function. The result of the computation is

$$(K_0{}^2 + K^2)\,\hat{G}(\mathbf{K}K_0;\mathbf{r}',t') = 2(2\pi)^{-1/2}\exp(-K_0\,t)\exp(iK_l\,x^l)\,H(t') \tag{6.2}$$

in which $K^2 = K_1{}^2 + K_2{}^2 + K_3{}^2$; $K_l x^l$ is summed over l for $l = 1, 2, 3$; $H(t)$ is Heaviside's step function

$$\begin{aligned} H(t) &= 1 \qquad (t > 0) \\ &= 0 \qquad (t < 0) \end{aligned}$$

and

$$\hat{G}(\mathbf{K},K_0;\mathbf{r}',t') = (2\pi)^{-3/2}\int_0^\infty dt\exp(-K_0\,t)\iiint_{-\infty}^{\infty} d^3x\exp(iK_l\,x^l)\,G(\mathbf{r},t;\mathbf{r}',t') \tag{6.3}$$

The inverse is given by the standard procedure.

$$G(\mathbf{r},t;\mathbf{r}',t') = \frac{(2\pi)^{-3/2}}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dK_0 \times$$

$$\times \exp(K_0\,t)\iiint_{-\infty}^{\infty} d^3K\exp(-iK_l\,x^l)\,\hat{G}(\mathbf{K},K_0;\mathbf{r}',t') \tag{6.4}$$

Placing into the integrand the value for the transformed Green's function given by equation (6.2), we find that

$$G(\mathbf{r},t;r',t') = \frac{H(t')\,H(t-t')\,\delta(t-t'-R)}{R} \tag{6.5}$$

where $R = |\mathbf{r}-\mathbf{r}'|$ is the distance between the field point situated at $\mathbf{r}$ and the position, $\mathbf{r}'$, of the source. Equation (6.5) determines that portion of the Green's function for which $t > 0$. To obtain the remainder of the function for which $t < 0$, we again resort to a one-sided Laplace transform, but over the range for t, $-\infty < t < 0$. The result of the calculation is

$$G(\mathbf{r}, t; \mathbf{r}', t') = H(-t')\,H(t' - t)\,\delta(t' - t - R)R \tag{6.6}$$

The addition of equations (6.5) and (6.6) yields the sought for Green's function.

We have accomplished what we had set out to do, and we should note, in passing, that the same procedure established the inverse of the Laplacian since it is a special case of the D'Alembertian which we obtain by setting t or K_0 equal to zero.

The answer in unique; we do not have the multiplicity of choice that conventional theory permits. We have stated the physical reason for our result several times, but it is worth reviewing. The introduction of field variables is merely another mathematical representation of the physical entities. Therefore, the new representation must be in one-to-one correspondence with the old one, and it is this requirement that determines the mathematical formalism.

Without the condition specified by the physical situation, there is no reason to seek out, as we did, the special Green's function which contained no part of a solution to the homogeneous equation. Thus, in conventional theory the Fourier transform of all the space-time variables is used. When this is done we obtain $K_0{}^2 - K^2$ instead of $K_0{}^2 + K^2$ and then the Green's function is not unique. In fact, it is at this point that the Dirac delta function, or a general distribution function, can be introduced.

Returning to the subject of the inverse operators, we would like to point out that the existence of the inverse assures the one-to-one correspondence between the source (particle) and the field (wave). Another way of stressing this equivalence is to say, that there is no particle-wave dualism, but only different mathematical representations for the same physical entity.

7. *Discussion*

A theoretical web has been woven which contains many elements that can be viewed best by considering each of them independently.

As the first thread, we had a study of the procedure which introduced the field concept into Newton's theory of gravitation. It followed so closely the same development of Maxwell's electromagnetic theory that the two theories were developed in parallel. Thus, what was once thought to be distinctive of electromagnetic theory was found to be valid for gravitational theory.

In the process of development, we achieved a generalization of the classical theory of gravitation. We found: (1) a relativistically covariant set of equations, (2) that gravitational effects are propagated with the speed of light, (3) the areas of similarity and difference between electromagnetic theory and gravitational theory, and (4) conservation laws which brought to light a force density, the dual of the Newtonian gravitational force.

A second thread which was woven into the web of our discourse was the tautological nature of both field theories. By the tautological nature of these theories is meant the equivalence of both sides of the equations. The equations function much like a dictionary in which the particle description of the source is translated into a field description. The particle-field dualism are not different aspects of the same physical entity but rather different mathematical representations of the same physical entity.

A third thread was the development of the differences between the two theories. These arose, as was to be expected, out of the inherently different physical content of the two theories. Thus, we found that electromagnetic theory required two field variables **E** and **H**, whereas gravitational theory, as far as present experiments and theory dictate, require but one field variable, **N**. The mathematical formalism nevertheless remains strikingly the same so that with slight modifications some of the results of electromagnetic theory can be used directly to illuminate gravitational theory. For example, the existence of gravitational waves is at once established. This result is not too surprising, because the present interpretation emphasizes the particle-field relation. For more than verbal assurance that gravitational waves exist, one need go no farther back than to equation (5.2). The solution to this equation for a particle source is given by the Lienard-Weichert solution for the electric field transcribed into gravitational terms. Thus, we can write

$$\mathbf{N}(\mathbf{r}, t) = \mu\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{(1-\mathbf{n}.\boldsymbol{\beta})^3 R^2}\right] + \frac{\mu}{c}\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\mathbf{n}.\boldsymbol{\beta})^3 R}\right] \tag{7.1}$$

in which $\mathbf{n} = \mathbf{R}/R$ is a unit vector directed from the charge to the field point (observation point). The velocity of the source particle is $c\boldsymbol{\beta}$, and $c\dot{\boldsymbol{\beta}}$ is the acceleration of the particle. $R = |\mathbf{r} - \mathbf{r}'|$ is the distance from the source to the field point. All the quantities in the brackets have to be evaluated at the retarded times. The manner in which Newton's law of gravitation has been generalized is clear. If the source is stationary, then equation (7.1) reduces to the value

$$\mathbf{N}(\mathbf{r}) = \frac{\mu\mathbf{r}}{r^3}$$

which is what would be expected. The general result, equation (7.1), exhibits the near and far field components which one readily associates with the static and radiation fields, respectively. The absence of the field variable, **M**, implies that $\mathbf{n} \times \mathbf{N} = 0$. (The same relation for electromagnetic theory yields $\mathbf{H} = \mathbf{n} \times \mathbf{E}$.) It follows that the gravitational field, as formulated, is a longitudinal wave field in which $\boldsymbol{\beta}$, $\dot{\boldsymbol{\beta}}$ and $\mathbf{n}$ are coplanar. We must leave to the applications, to be reported later, the exploitation of the gravitational field equations.

We have mentioned that the force density dual to the Newtonian force accounts for the precession of the perihelion of the orbit of one particle gravitating about another. The details of this computation will be submitted soon.

The final thread of our discourse was not made explicit, and it should be. The analysis we made of Newton's law of gravitation and Coulomb's law of force factored each of them into two elements. In one instance, we considered the factored elements particles and, in another, one element was a particle and the other a field variable. The factorization, of course, is not unique. Yet, the specific choice, as we saw, determined the mathematical formalism. The non-uniqueness of the factorization implies that there are many formalisms which can be used, but they must be equivalent since they are representing the same physical event. Therefore, the elements of one type of factorization must be *replacements* for the elements of another type of division. The identification between the elements of the different factorizations constitutes the reason for the tautological nature of the resulting relations and for the need to represent them with sufficient care.

The physical basis for what we have just discussed is rooted in the measuring process. Every experiment consists of a system to be measured and a detector. The data collected is clearly joint property. To associate the data solely with one or the other element in an experiment has neither physical nor logical justification. We would enunciate this result as a principle: it is the interaction between the elements, and not the elements themselves, which are the physical observables.

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